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On the cross-section distortion of thin-walled beams with multi-cell cross-sections subjected to bending

R. Pavazza *, B. Blagojević

Faculty of Electrical Engineering, Mechanical Engineering and Naval Architecture, University of Split, Ruđera Boskovića bb, 21000 Split, Croatia

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Abstract

This paper deals with distortion of the cross-section contour of thin-walled beams with simple multi-cell closed rectangular cross-sections. The cross-section distortion is considered in the limit case. It is assumed that beam plates are hinged together along their longitudinal edges. Double symmetric three and two-cell closed cross-sections are considered. The stresses and displacements are obtained in the closed analytical form. The additional stresses and displacements due to distortion with respect to the stresses and displacements of the ordinary theory of bending are obtained. The boundary conditions are given in the general form. Some illustrative examples are given.

Keywords: Distortion of cross-sections; Bending of thin-walled beams; Three and two cell closed sections; Hinged walls; Analytic method

1. Introduction

The reliability of the assumption, in the theory of bending of thin-walled beams, that the shape of the cross-section is maintained depends on the stiffness of the beam transverse framing, or the cross-section itself (Vlasov, 1961); in some cases, also on the load distribution in the transverse direction (Kollbrunner and Basler, 1969).

The thin-walled beams are assembled of a number of thin plates that are restrained along their longitudinal edges. The plates can be stiffened by frames, usually in the transverse direction; sometimes, also by transverse bulkheads ("diaphragms"). Transverse framing, or the cross-section itself, cannot prevent the

* Corresponding author. Tel.: +38 521 305 868; fax: +385 214 63 877.
E-mail addresses: radoslav.pavazza@fesb.hr (R. Pavazza), branko.blagojevic@fesb.hr (B. Blagojević).

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cross-section distortion entirely. In the limit, it may be assumed that plates are "hinged" along their longitudinal edges.

These two different types of structural behaviour (with the "rigid" cross-sections and hinged cross-sections) may be considered as two limiting cases of stresses and deformations of the actual structure (Kollbrunner and Basler, 1969).

The theory that deals with such idealized beams, with hinged plates, is the folded plate theory (Schwyzer, 1920; Kollbrunner and Basler, 1969). By the theory, the longitudinal deformations and therefore the longitudinal normal stresses are equal at hinged connections. Hence, the normal stresses are linearly distributed over the cross-sections of each plate. The end cross-sections are assumed rigid. Each plate is at most preceded and followed by one plate only; the method is no longer applicable if one hinge belongs to three or more plates.

The problem of the cross-section distortion can be solved in an "exact" way, in general, by using three-dimensional models (Hughes, 1983; Bull, 1988). However, in the case of ordinary beams, with small dimensions of the cross-section contour with respect to the length of the beam, simpler analytical as well as numerical methods may also be applied (Boitzov, 1972; Boswell and Zhang, 1984; Boswell and Li, 1995; Hsu et al., 1995; Kim and Kim, 1999, 2000a,b, 2001; Pavazza and Matoković, 2000; Kim et al., 2002; Pavazza, 2002).

In this paper, an analytical approach to the problem of the cross-section distortion of prismatic beams with closed rectangular thin-walled cross-sections subjected to bending with influence of shear will be considered. The double symmetric cross-sections with three and two closed cells will be analysed. The results will be given in the analytical closed form, suitable for parametric studies.

2. Thin-walled beams subjected to bending with influence of shear

The forces–displacements relations for a prismatic thin-walled beam subjected to bending, by ordinary beam theories that include the shear influence (Filin, 1975), are governed by the following differential equations

$$\frac{\mathrm{d}w}{\mathrm{d}x} = -\beta, \quad EI_y\frac{\mathrm{d}^2w}{\mathrm{d}x^2} = -M_y, \quad EI_y\frac{\mathrm{d}^3w}{\mathrm{d}x^3} = -\frac{\mathrm{d}M_y}{\mathrm{d}x} = -Q_z, \quad EI_y\frac{\mathrm{d}^4w}{\mathrm{d}x^4} = -\frac{\mathrm{d}^2M_y}{\mathrm{d}x^2} = -\frac{\mathrm{d}Q_z}{\mathrm{d}x} = q_z, \tag{1}$$

where $w = w(x)$ is the displacement in the z-direction due to bending, $\beta = \beta(x)$ is the angular displacement with respect to the y-axis due to bending, I_y is the cross-section moment of inertia with respect to the y-axis, $M_y = M_y(x)$ is the bending moment with respect to the y-axis, $Q_z = Q_z(x)$ is the shearing force with respect to the z-axis, $q_z = q_z(x)$ is the line load per unit length in the z-direction, $Oxyz$ is the rectangular co-ordinate system, where the y and z-axis coincide with the principal axes, where the z-axis is the axis of symmetry, and E is the modulus of elasticity (Fig. 1);

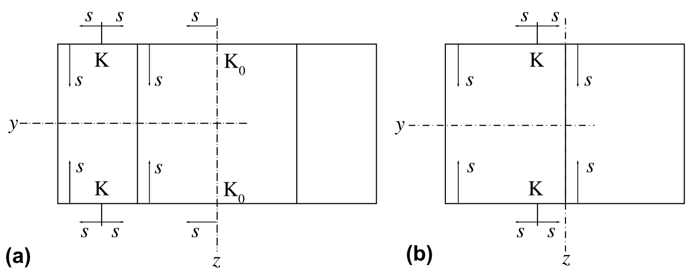

Fig. 1. Rectangular closed thin-walled cross-sections with coordinate systems: (a) three cell section; (b) two cell-section.

$$\frac{dw_s}{dx} = -\beta_s, \quad GA_s\frac{dw_s}{dx} = Q_z, \quad GA_s\frac{d^2w_s}{dx^2} = \frac{dQ_z}{dx} = -q_z, \tag{2}$$

where $w_s = w_s(x)$ is the displacement in the z-direction due to shear, $\beta_s = \beta_s(x)$ is the angular displacement with respect to the y-axis due to shear, A_s is the cross-section shear area;

$$w_t = w + w_s, \quad \beta_t = \beta + \beta_s, \tag{3}$$

where w_t is the total displacement in the z-direction and β_t is the total angular displacement with respect to the y-axis.

The stresses are given by

$$\sigma_x = \frac{M_y}{I_y}z, \quad \tau_{xs} = \frac{T_{xs}}{t}, \quad T_{xs} = \frac{Q_z}{I_y}S_y^*, \tag{4}$$

where $\sigma_x = \sigma_x(x,s)$ is the normal stress in the x-direction, $\tau_{xs} = \tau_{xs}(x,s)$ is the shear stress in the s-direction, $T_{xs} = T_{xs}(x,s)$ is the shear flow in the s-direction, $z = z(s)$ is the cross-section rectangular co-ordinate, $S_y^* = S_y^*(s)$ is the statical moment of the "cut-off portion" of the cross-section with respect to the y-axis, $t = t(s)$ is the wall thickness and s is the cross-section curvilinear co-ordinate, with the origins K_0 and K, where $T_{xs} = 0$ (Fig. 1);

It is assumed that the shear influence on bending is small; it will be taken into account in the calculations of displacements only, by using (2) and (3). The cross-section contour is assumed "rigid" (neglecting the contour contraction).

Thus, from Hooke's law,

$$\sigma_x = E\varepsilon_x, \tag{5}$$

where $\varepsilon_x = \varepsilon_x(x,s)$ is the strain in x-direction. The shear flow is defined by the normal stress σ_x, by the equilibrium equation in the x-direction

$$\frac{\partial\sigma_x}{\partial x}t + \frac{\partial T_{xs}}{\partial s} = 0. \tag{6}$$

The normal stress in the s-direction $\sigma_s = \sigma_s(x,s)$, which is ignored in Eq. (5) as a small quantity, can be obtained by the shear flow, by using the equilibrium equation in the s-direction

$$\frac{\partial T_{sx}}{\partial x} + \frac{\partial(\sigma_s t)}{\partial s} = 0, \tag{7}$$

where $T_{sx} = T_{xs}$.

3. Thin-walled beams subjected to bending with the cross-section distortion

3.1. Distortion of the cross-sections of thin-walled beams subjected to bending

If the frames or diaphragms are omitted, or the cross-section is not stiff by itself, the cross-sections of thin-walled beams are no longer forced to maintain their shape. In the limit, it may be assumed that the beam walls are hinged along their longitudinal edges. Thus, the walls will be loaded by forces per unit length along their longitudinal edges, in their planes (Pavazza, 2002).

In the case of symmetrical closed thin-walled cross-sections with three cells (Fig. 2a), the inner vertical walls and the central horizontal walls may be treated due to symmetry as a unique beam with the rigid cross-section (beam component 1). Two cell cross-sections may be treated as a special case of three cell cross-sections (Fig. 2b).

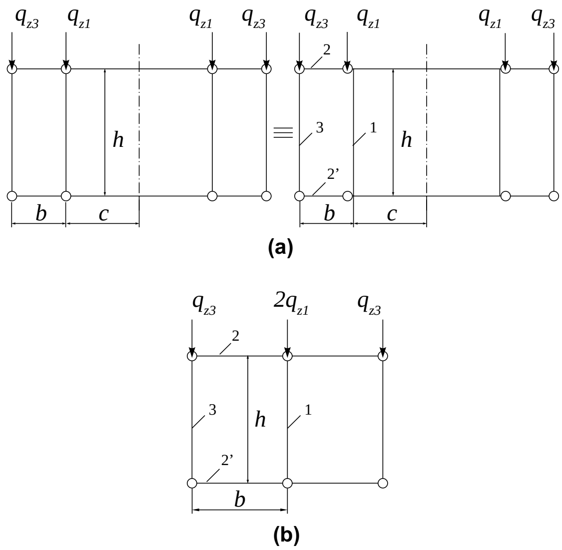

Fig. 2. Rectangular closed thin-walled sections with hinged walls: (a) three cell section; (b) two cell-section.

3.2. Beam components

The beam may be decomposed into four beam components (Fig. 3). In the case of double symmetric beam cross-sections, it may be assumed that the beam components 1 and 3 are subjected to bending with influence of shear and the beam components 2 and $2'$ to tension and shearing due to distortion of the cross-section shape.

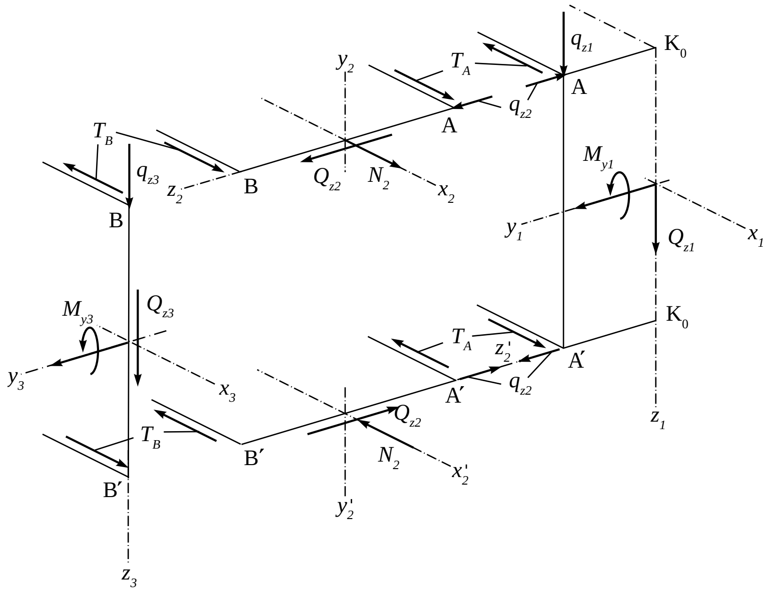

Fig. 3. Beam components of the beam with double symmetrical three cell thin-walled cross-section.

For the beam components 1 and 3, the following differential equations may be written ($i = 1, 3$) (Pavazza, 1991)

$$\frac{dw_i}{dx} = -\beta_i, \quad EI_{yi}\frac{d^2w_i}{dx^2} = -M_{yi}, \quad EI_{yi}\frac{d^3w_i}{dx^3} = -\frac{dM_{yi}}{dx} = -(Q_{zi} - m_{yi}),$$
$$EI_{yi}\frac{d^4w_i}{dx^4} = -\frac{d^2M_{yi}}{dx^2} = -\frac{d}{dx}(Q_{zi} - m_{yi}) = q_{zi} + \frac{dm_{yi}}{dx}, \tag{8}$$

where $w_i = w_i(x)$ is the displacement in the z_i-direction, $\beta_i = \beta_i(x)$ is the angular displacement with respect to the y_i-axis, I_{yi} is the cross-section moment of inertia with respect to the y_i-axis, $M_{yi} = M_{yi}(x)$ is the bending moment with respect to the y_i-axis, $Q_{zi} = Q_{zi}(x)$ is the shearing force with respect to the z_i-axis, $q_{zi} = q_{zi}(x)$ is the force per unit length in z_i-direction, $m_{yi} = m_{yi}(x)$ is the moment per unit length, $O_ix_iy_iz_i$ is the rectangular co-ordinate system, where the y_i, z_i-axes coincide with principal axes of the corresponding cross-sections;

$$\frac{dw_{si}}{dx} = -\beta_{si}, \quad GA_{si}\frac{dw_{si}}{dx} = Q_{zi}, \quad GA_{si}\frac{d^2w_{si}}{dx^2} = \frac{dQ_{zi}}{dx} = -q_{zi}, \tag{9}$$

where $w_{si} = w_{si}(x)$ is the displacement in the z_i-direction due to shear, $\beta_{si} = \beta_{si}(x)$ is the angular displacement with respect to the y_i-axis due to shear, A_{si} is the cross-section shear area;

$$w_{ti} = w_i + w_{si}, \quad \beta_{ti} = \beta_i + \beta_{si}, \tag{10}$$

where w_{ti} is the total displacement in the z_i-direction and β_{ti} is the total angular displacements with respect to the y_i-axis.

For the beam component 1, the moment per unit length may be expressed as

$$m_{y1} = T_Ah, \tag{11}$$

where $T_A = T_A(x)$ is the force per unit length along the longitudinal edge A and h is the height of the cross-section. The stresses may be expressed as follows

$$\sigma_{x1} = \frac{M_{y1}}{I_{y1}}z_1, \quad \tau_{xs1} = \frac{T_{xs1}}{t_1}, \quad T_{xs1} = \frac{(Q_{z1} - m_{y1})}{I_{y1}}S^*_{y1} + T_A|_{A-A'}. \tag{12}$$

In this equation T_A is assumed to be distributed along the vertical wall of the beam components 1 only, as a constant (which is expressed symbolically by a vertical line); it represents the influence of the beam components 2 on the beam component 1.

For the beam component 3, similarly, one has

$$m_{y3} = T_Bh, \tag{13}$$

where $T_B = T_B(x)$ is the force per unit length along the longitudinal edge B;

$$\sigma_{x3} = \frac{M_{y3}}{I_{y3}}z_3, \quad \tau_{xs3} = \frac{T_{xs3}}{t_3}, \quad T_{xs3} = \frac{(Q_{z3} - m_{y3})S^*_{y3}}{I_{y3}} + T_B. \tag{14}$$

Here is assumed that the origin of s is at B, i.e B' (Fig. 1).

For the beam component 2, it may be written (Pavazza et al., 2001)

$$EA_2\frac{du_{2p}}{dx} = N_2, \quad EA_2\frac{d^2u_{2p}}{dx^2} = \frac{dN_2}{dx} = -q_{x2}, \tag{15}$$

where

$$q_{x2} = T_A + T_B, \tag{16}$$

where A_2 is the cross-section area of the beam component 2, $u_{2p} = u_{2p}(x)$ is the displacement of the cross-section in the x_2-direction as a plane section and $N_2 = N_2(x)$ is the normal force;

$$Q_{z2} = m_{y2}, \quad \frac{\mathrm{d}Q_{z2}}{\mathrm{d}x} = -q_{z2} = \frac{\mathrm{d}m_{y2}}{\mathrm{d}x}, \tag{17}$$

where

$$m_{y2} = -(T_A - T_B)\frac{b}{2}. \tag{18}$$

Then

$$u_2 = u_{2p} + f,$$

where $u_2 = u_2(x, z_2)$ is the displacement in the x_2-direction; $f = f(x, z)$ is warping due to shear

$$f = \frac{Q_{z2}}{GA_2} z_2 + \frac{q_{x2}}{A_2 E}\left(z_2^2 - \frac{b^2}{12}\right) + g, \tag{19}$$

where $g = g(x, z)$ is a function that can be obtained by assuming that $f = 0$ when $w_1 = w_3$, when according to (4)

$$T_A - T_B = k(T_A + T_B), \quad k = \frac{2z_{20}}{b} = \frac{S^*_{yA} - S^*_{yB}}{S^*_{yA} + S^*_{yB}}, \tag{20}$$

where z_{20} is the coordinate of the shear flow zero point for the rigid cross-section, S^*_{yA} and S^*_{yB} are the statical moments of the cut-off portion of the rigid cross-section for the points A and B, respectively.

Thus, from (19) and (20)

$$g = \frac{k(T_A + T_B)b}{2GA_2} z_2 - \frac{T_A - T_B}{kEA_2}\left(z_2^2 - \frac{b^2}{12}\right), \tag{21}$$

i.e.

$$f = [T_A - T_B - k(T_A + T_B)]\frac{b^2}{2GA_2}\left[\frac{2G}{kE}\left(\frac{z_2}{b}\right)^2 + \frac{z_2}{b} - \frac{G}{6kE}\right], \tag{22}$$

and

$$u_2 = u_{2p} - [T_A - T_B - k(T_A + T_B)]\frac{b^2}{2GA_2}\left[\frac{2G}{kE}\left(\frac{z_2}{b}\right)^2 + \frac{z_2}{b} - \frac{G}{6kE}\right]. \tag{23}$$

The stresses then read

$$\begin{aligned} \sigma_{x2} &= E\varepsilon_x = \frac{N_2}{A_2} - \frac{Eb^2}{2GA_2}\left[\frac{2G}{kE}\left(\frac{z_2}{b}\right)^2 + \frac{z_2}{b} - \frac{G}{6kE}\right]\frac{\mathrm{d}}{\mathrm{d}x}[T_A - T_B - k(T_A + T_B)], \\ \tau_{xz2} &= \frac{T_{xz2}}{t_2}, \quad T_{xz2} = -\frac{1}{2}(T_A - T_B) + (T_A + T_B)\frac{z_2}{b}, \end{aligned} \tag{24}$$

where $\varepsilon_x = \frac{\partial u_2}{\partial x}$.

The stresses given by (24) satisfy the equilibrium equation given by (6) only if

$$\frac{\mathrm{d}}{\mathrm{d}x}[T_A - T_B - k(T_A + T_B)] = \text{const}, \tag{25}$$

otherwise the solution can be used only approximately.

The beam component 2′ will be under antisymmetric tension and shearing, with respect to the beam component 2.

3.3. Compatibility conditions

The compatibility conditions, along the edges A and B, taking into account (23), may be expressed as

$$\begin{aligned}
-\beta_1\frac{h}{2} &= u_{2p} + \frac{b^2}{4GA_2}\left(1-\frac{2G}{3kE}\right)[T_A - T_B - k(T_A+T_B)],\\
-\beta_3\frac{h}{2} &= u_{2p} - \frac{b^2}{4GA_2}\left(1+\frac{2G}{3kE}\right)[T_A - T_B - k(T_A+T_B)].
\end{aligned} \tag{26}$$

Hence

$$\begin{aligned}
u_{2p} &= -(\beta_1+\beta_3)\frac{h}{4} + \frac{b^2}{6EA_2k}[T_A - T_B - k(T_A+T_B)],\\
T_A - T_B - k(T_A+T_B) &= -(\beta_1-\beta_3)\frac{GA_2h}{b^2}.
\end{aligned} \tag{27}$$

Then, referring to the first equation of (15) and the first and second equations of (8)

$$N_2 = -\left(\frac{M_{y1}}{I_{y1}}+\frac{M_{y3}}{I_{y3}}\right)\frac{A_2h}{4} - \left(\frac{M_{y1}}{I_{y1}}-\frac{M_{y3}}{I_{y3}}\right)\frac{GA_2h}{6kE}, \tag{28}$$

and, referring to the second equation of (15), (16), and the third equation of (8)

$$T_A+T_B = \left(\frac{Q_{z1}-m_{y1}}{I_{y1}}+\frac{Q_{z3}-m_{y3}}{I_{y3}}\right)\frac{A_2h}{4} + \left(\frac{Q_{z1}-m_{y1}}{I_{y1}}-\frac{Q_{z3}-m_{y3}}{I_{y3}}\right)\frac{GA_2h}{6kE}, \tag{29}$$

i.e.

$$T_A+T_B = -\frac{EA_2h}{4}\times\frac{\mathrm{d}^3}{\mathrm{d}x^3}(w_1+w_3) - \frac{GA_2h}{6k}\times\frac{\mathrm{d}^3}{\mathrm{d}x^3}(w_1-w_3). \tag{30}$$

The second equation of (27), taking into account the first equation of (8), may be written as

$$T_A - T_B + k(T_A+T_B) = -\frac{GA_2h}{b^2}\times\frac{\mathrm{d}}{\mathrm{d}x}(w_1-w_3). \tag{31}$$

The functions T_A and T_B may then be obtained from (30) and (31):

$$\begin{aligned}
T_A &= -\frac{EA_2h}{8}(1+k)\frac{\mathrm{d}^3}{\mathrm{d}x^3}(w_1+w_3) - \frac{GA_2h}{12}\left(1+\frac{1}{k}\right)\frac{\mathrm{d}^3}{\mathrm{d}x^3}(w_1-w_3) + \frac{GA_2h}{2b^2}\frac{\mathrm{d}}{\mathrm{d}x}(w_1-w_3),\\
T_B &= -\frac{EA_2h}{8}(1-k)\frac{\mathrm{d}^3}{\mathrm{d}x^3}(w_1+w_3) + \frac{GA_2h}{12}\left(1-\frac{1}{k}\right)\frac{\mathrm{d}^3}{\mathrm{d}x^3}(w_1-w_3) - \frac{GA_2h}{2b^2}\frac{\mathrm{d}}{\mathrm{d}x}(w_1-w_3).
\end{aligned} \tag{32}$$

For the case $w_1 = w_3$, one obtains

$$T_A = -\frac{EA_2h}{4}(1+k)\frac{\mathrm{d}^3w_1}{\mathrm{d}x^3},\quad T_B = -\frac{EA_2h}{4}(1-k)\frac{\mathrm{d}^3w_1}{\mathrm{d}x^3}. \tag{33}$$

Substitution of (33) into (8), taking into account (11) and (13), yields

$$EI_{yc}\frac{\mathrm{d}^4w_1}{\mathrm{d}x^4} = q_{z1},\quad EI_{ys}\frac{\mathrm{d}^4w_3}{\mathrm{d}x^4} = q_{z3}, \tag{34}$$

where

$$I_{yc} = I_{y1} + (1+k)\frac{A_2 h^2}{4}, \quad I_{ys} = I_{y3} + (1-k)\frac{A_2 h^2}{4}. \tag{35}$$

Since

$$q_{z1} + q_{z3} = \frac{q_z}{2}, \quad I_{yc} + I_{ys} = \frac{1}{2} I_y, \tag{36}$$

from (34), taking into account (1), one obtains $w_1 = w_3 = w$, i.e.

$$\frac{q_{z1}}{I_{yc}} = \frac{q_{z3}}{I_{ys}} = \frac{q_z}{I_y}. \tag{37}$$

Thus, referring to (1) and (4), T_A and T_B, given by (32), become

$$T_A = \frac{Q_z}{I_y} S^*_{yA} = -T_{xsA}, \quad T_B = \frac{Q_z}{I_y} S^*_{yB} = T_{xsB}, \tag{38}$$

where

$$S^*_{yA} = \frac{A_2 h}{4}(1+k), \quad S^*_{yB} = \frac{A_2 h}{4}(1-k). \tag{39}$$

3.4. Internal forces and displacements

By substituting (28) into the fourth equation of (8), taking onto account (11) and (13), the following differential equations may be written

$$\begin{aligned} &EI_{yc}\frac{d^4 w_1}{dx^4} - \frac{EA_2 h^2}{8}(1+k)\left(1 - \frac{2G}{3kE}\right)\frac{d^4}{dx^4}(w_1 - w_3) - \frac{GA_2 h^2}{2b^2} \times \frac{d^2}{dx^2}(w_1 - w_3) = q_{z1}, \\ &EI_{ys}\frac{d^4 w_1}{dx^4} + \frac{EA_2 h^2}{8}(1-k)\left(1 + \frac{2G}{3kE}\right)\frac{d^4}{dx^4}(w_1 - w_3) + \frac{GA_2 h^2}{2b^2} \times \frac{d^2}{dx^2}(w_1 - w_3) = q_{z3}, \end{aligned} \tag{40}$$

where I_{yc} and I_{ys} are given by (35).

By multiplying the first equation from (36) by $I_{ys}/(I_{yc} + I_{ys})$ and the second by $I_{yc}/(I_{yc} + I_{ys})$, and subtracting the second equation from the first, the following differential equation may be written

$$EI_{ycs}\frac{d^4 w_{1-3}}{dx^4} - k_\beta \frac{d^2 w_{1-3}}{dx^2} = q_{zcs}, \tag{41}$$

where

$$\begin{aligned} &w_{1-3} = w_1 - w_3, \quad q_{zcs} = \frac{q_{z1} I_{ys} - q_{z3} I_{yc}}{I_{yc} + I_{ys}}, \quad k_\beta = \frac{GA_2}{2}\left(\frac{h}{b}\right)^2, \\ &I_{ycs} = \frac{I_{yc} I_{ys}}{I_{yc} + I_{ys}}\left[1 - \frac{(1+k)\left(1 - \frac{2G}{3kE}\right) I_{ys} + (1-k)\left(1 + \frac{2G}{3kE}\right) I_{yc}}{I_{yc} I_{ys}} \times \frac{A_2 h^2}{8}\right]. \end{aligned} \tag{42}$$

The following differential equations can then be plotted (see Appendix A)

$$\begin{aligned} &\frac{dw_{1-3}}{dx} = -\beta_{1-3}, \quad EI_{ycs}\frac{d^2 w_{1-3}}{dx^2} = -M_{ycs}, \quad EI_{ycs}\frac{d^3 w_{1-3}}{dx^3} = -\frac{dM_{ycs}}{dx} = -(Q_{zcs} - m_{ycs}), \\ &EI_{ycs}\frac{d^4 w_{1-3}}{dx^4} = -\frac{d^2 M_{ycs}}{dx^2} = -\frac{d}{dx}(Q_{zcs} - m_{ycs}) = q_{zcs} + \frac{dm_{ycs}}{dx}, \end{aligned} \tag{43}$$

where

$$m_{ycs} = k_\beta \frac{\mathrm{d}w_{1-3}}{\mathrm{d}x} = -k_\beta \beta_{1-3}. \tag{44}$$

Taking into account the first equation of (43), it may be written

$$\frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 w_3}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 w_{1-3}}{\mathrm{d}x^2}, \tag{45}$$

i.e. referring to the second equation of (8) and the second equation of (43)

$$\frac{M_{y1}}{I_{y1}} - \frac{M_{y3}}{I_{y3}} = \frac{M_{ycs}}{I_{ycs}}. \tag{46}$$

From the equilibrium, it follows

$$M_{y1} + M_{y3} - N_2 h = \frac{1}{2} M_y, \quad Q_{z1} + Q_{z3} = \frac{1}{2} Q_z. \tag{47}$$

Substitution of (28) into the first equation of (47) yields

$$\frac{M_{y1}}{I_{y1}} I'_{yc} + \frac{M_{y3}}{I_{y3}} I'_{ys} = \frac{1}{2} M_y, \tag{48}$$

where

$$I'_{yc} = I_{y1} + \frac{A_2 h^2}{4}\left(1 + \frac{2G}{3kE}\right), \quad I'_{ys} = I_{y3} + \frac{A_2 h^2}{4}\left(1 - \frac{2G}{3kE}\right). \tag{49}$$

Here

$$I'_{yc} + I'_{ys} = \frac{1}{2} I_y. \tag{50}$$

From (46) and (48), taking into account (50), it may be written

$$M_{y1} = \frac{M_y}{I_y} I_{y1} + 2\frac{I'_{ys}}{I_{ycs}} \times \frac{M_{ycs}}{I_y} I_{y1}, \quad M_{y3} = \frac{M_y}{I_y} I_{y3} - 2\frac{I'_{yc}}{I_{ycs}} \times \frac{M_{ycs}}{I_y} I_{y3}, \tag{51}$$

and according to third equation of (8) and the third equation of (43)

$$\begin{aligned} Q_{z1} - m_{y1} &= \frac{Q_z}{I_y} I_{y1} + 2\frac{I'_{ys}}{I_{ycs}} \times \frac{Q_{zcs} - m_{ycs}}{I_y} I_{y1}, \\ Q_{z3} - m_{y3} &= \frac{Q_z}{I_y} I_{y3} - 2\frac{I'_{yc}}{I_{ycs}} \times \frac{Q_{zcs} - m_{ycs}}{I_y} I_{y3}. \end{aligned} \tag{52}$$

According to (51), taking into account the second equations of (8), (1) and (43), it may be written

$$\begin{aligned} w_1 &= w + \Delta w_1, \quad w_3 = w + \Delta w_3, \\ \beta_1 &= \beta + \Delta\beta_1, \quad \beta_3 = \beta + \Delta\beta_3, \end{aligned} \tag{53}$$

where

$$\begin{aligned} \Delta w_1 &= 2\frac{I'_{ys}}{I_y} w_{1-3}, \quad \Delta w_3 = -2\frac{I'_{yc}}{I_y} w_{1-3}, \\ \Delta\beta_1 &= 2\frac{I'_{ys}}{I_y} \beta_{1-3}, \quad \Delta\beta_3 = -2\frac{I'_{yc}}{I_y} \beta_{1-3}. \end{aligned} \tag{54}$$

From (28) and (51), one has

$$N_2 = -\frac{M_y}{I_y} \times \frac{A_2 h}{2} + \frac{I'_{yc} - I'_{ys}}{I_{ycs}} \times \frac{M_{ycs}}{I_y} \times \frac{A_2 h}{2} - \frac{I'_{yc} + I'_{ys}}{I_{ycs}} \times \frac{M_{ycs}}{I_y} \times \frac{G A_2 h}{3kE}. \tag{55}$$

From (53), it follows

$$w = \frac{1}{2}(w_1 + w_3) + \frac{I'_{yc} - I'_{ys}}{I_y} w_{1-3}. \tag{56}$$

Thus, taking into account the third equations of (1), (41) and (43), T_A and T_B given by (32) may be written as

$$\begin{aligned} T_A &= \frac{Q_z}{I_y} S^*_{yA} - \frac{Q_{zcs}}{I_y} \times \frac{I'_{yc} - I'_{ys}}{I_{ycs}} S^*_{yA} - k_\beta \frac{\beta_{1-3}}{h}, \\ T_B &= \frac{Q_z}{I_y} S^*_{yB} - \frac{Q_{zcs}}{I_y} \times \frac{I'_{yc} - I'_{ys}}{I_{ycs}} S^*_{yB} + k_\beta \frac{\beta_{1-3}}{h}, \end{aligned} \tag{57}$$

where S^*_{yA} and S^*_{yB} are given by (39).

By substituting (57) into (52), taking into account (12) and (14), (35), (39) and (49), where

$$I_{y1} + S^*_{yA} h = I_{yc}, \quad I_{y3} + S^*_{yB} h = I_{ys}, \quad \frac{2I'_{ys} I_{y1} + (I'_{yc} - I'_{ys}) S^*_{yA} h}{I_y} = \frac{2I'_{yc} I_{y3} + (I'_{yc} - I'_{ys}) S^*_{yB} h}{I_y} = I_{ycs},$$

the following relation can be obtained

$$Q_{z1} = \frac{Q_z}{I_y} I_{yc} + Q_{zcs}, \quad Q_{z3} = \frac{Q_z}{I_y} I_{ys} - Q_{zcs}. \tag{58}$$

Hence, taking into account the second equation of (47)

$$Q_{zcs} = \frac{Q_{z1} I_{ys} - Q_{z3} I_{yc}}{I_{yc} + I_{ys}}. \tag{59}$$

The displacements due to shear w_s and w_{si} may be obtained, according to the second equations of (2) and (9), as follows

$$w_s = \frac{M_y - M_y^{(A)}}{G A_s}, \quad w_{s1} = \frac{M_{y1} - M_{y1}^{(A)}}{G A_{s1}}, \quad w_{s3} = \frac{M_{y3} - M_{y3}^{(A)}}{G A_{s3}}, \tag{60}$$

where $M_y^{(A)}$, $M_{y1}^{(A)}$ and $M_{y3}^{(A)}$ are the bending moments of the beam and the beam components 1 and 3 at the beam left end. From (60) and (51), it follows

$$\begin{aligned} w_{s1} &= \frac{M_y - M_y^{(A)}}{G A_{s1}} \times \frac{I_{y1}}{I_y} + \frac{M_{ycs} - M_{ycs}^{(A)}}{G A_{s1}} \times 2 \frac{I'_{ys} I_{y1}}{I_{ycs} I_y}, \\ w_{s3} &= \frac{M_y - M_y^{(A)}}{G A_{s3}} \times \frac{I_{y3}}{I_y} - \frac{M_{ycs} - M_{ycs}^{(A)}}{G A_{s3}} \times 2 \frac{I' I_{y3}}{I_{ycs} I_y}. \end{aligned} \tag{61}$$

The unknown shear areas A_{s1} and A_{s3} can be obtained from the condition: when $w_{s1} = w_{s3} = w_s$, then $w_1 = w_3 = w$, $M_{ycs} = 0$. Thus, from (61) and (60)

$$A_{s1} = \frac{I_{y1}}{I_y} A_s, A_{s3} = \frac{I_{y3}}{I_y} A_s. \tag{62}$$

By substituting (62) into (61), one may finally write

$$w_{s1} = w_s + \Delta w_{s1}, \quad w_{s3} = w_s + \Delta w_{s3}, \tag{63}$$

where

$$\Delta w_{s1} = 2\frac{I'_{ys}}{I_{ycs}} \times \frac{M_{ycs} - M_{ycs}^{(A)}}{GA_s}, \quad \Delta w_{s3} = -2\frac{I'_{yc}}{I_{ycs}} \times \frac{M_{ycs} - M_{ycs}^{(A)}}{GA_s}. \tag{64}$$

3.5. Stresses

The stresses for the beam component 1 may finally be obtained by substituting the first equation of (51) and the first equation of (52) into (12):

$$\sigma_{x1} = \frac{M_{y1}}{I_y} z_1 + \Delta\sigma_{x1}, \quad T_{xs1} = \frac{Q_z}{I_y} S_{y1}^* + \frac{Q_z}{I_y} S_{yA}^* \bigg|_A^{A'} + \Delta T_{xs1}, \tag{65}$$

where

$$\begin{aligned} \Delta\sigma_{x1} &= \frac{2I'_{ys}}{I_{ycs}} \times \frac{M_{ycs}}{I_y} z_1, \\ \Delta T_{xs1} &= \frac{2I'_{ys}}{I_{ycs}} \times \frac{Q_{zcs} - m_{ycs}}{I_y} S_{y1}^* - \frac{Q_{zcs}}{I_y} \times \frac{I'_{yc} - I'_{ys}}{I_{ycs}} S_{yA}^* \bigg|_A^{A'} + \frac{m_{ycs}}{h}\bigg|_A^{A'}. \end{aligned} \tag{66}$$

The stresses for the beam component 3 may be obtained by substituting the second equation of (51) and the second equation of (52) into (14):

$$\sigma_{x3} = \frac{M_{y3}}{I_y} z_3 + \Delta\sigma_{x3}, \quad T_{xs3} = \frac{Q_z}{I_y} S_{y3}^* + \frac{Q_z}{I_y} S_{yB}^* + \Delta T_{xs3}, \tag{67}$$

where

$$\begin{aligned} \Delta\sigma_{x3} &= -\frac{2I'_{yc}}{I_{ycs}} \times \frac{M_{ycs}}{I_y} z_3, \\ \Delta T_{xs3} &= -\frac{2I'_{yc}}{I_{ycs}} \times \frac{Q_{zcs} - m_{ycs}}{I_y} S_{y3}^* - \frac{Q_{zcs}}{I_y} \frac{I'_{yc} - I'_{ys}}{I_{ycs}} S_{yB}^* - \frac{m_{ycs}}{h}. \end{aligned} \tag{68}$$

The stresses for the beam component 2 may be obtained by substituting (55) and (57) into (24), taking into account the first equations of (1) and (43), and (39):

$$\sigma_{x2} = -\frac{M_y}{I_y} \times \frac{h}{2} + \Delta\sigma_{x2}, T_{xz2} = -\frac{Q_z}{2I_y}(S_{yA}^* - S_{yB}^*) + \frac{Q_z}{I_y}(S_{yA}^* + S_{yB}^*)\frac{z_2}{b} + \Delta T_{xz2}, \tag{69}$$

where

$$\begin{aligned} \Delta\sigma_{x2} &= \frac{I'_{yc} - I'_{ys}}{I_{ycs}} \times \frac{M_{ycs}}{I_y} \times \frac{h}{2} - \frac{I'_{yc} + I'_{ys}}{I_{ycs}} \times \frac{M_{ycs}}{I_y} \times \frac{Gh}{3kE} + \frac{I'_{yc} + I'_{ys}}{I_{ycs}} \times \frac{M_{ycs}}{I_y} \times \frac{z_2}{b} h \\ &\quad + \frac{I'_{yc} + I'_{ys}}{I_{ycs}} \times \frac{M_{ycs}}{I_y} \times \frac{Gh}{3kE}\left[6\left(\frac{z_2}{b}\right)^2 - \frac{1}{2}\right], \\ \Delta T_{xz2} &= \frac{Q_{zcs}}{2I_y} \times \frac{I'_{yc} - I'_{ys}}{I_{ycs}}(S_{yA}^* - S_{yB}^*) - \frac{Q_{zcs}}{I_y} \times \frac{I'_{yc} - I'_{ys}}{I_{ycs}}(S_{yA}^* + S_{yB}^*)\frac{z_2}{b} - \frac{m_{ycs}}{h}. \end{aligned} \tag{70}$$

The force per unit length q_{z2} can be obtained from (17) and (18), taking into account (57):

$$q_{z2} = -\frac{q_{z2}}{I_y}(S^*_{yA} - S^*_{yB})\frac{b}{2} + \Delta q_{z2}, \tag{71}$$

where

$$\Delta q_{z2} = \frac{q_{zcs}}{I_y} \times \frac{I'_{yc} - I'_{ys}}{I_{ycs}}(S^*_{yA} - S^*_{yB})\frac{b}{2} - \frac{M_{ycs}}{I_{ycs}} \times \frac{b}{h}k_\beta. \tag{72}$$

Here

$$\sigma_{y2A} = -\frac{q_{z2}}{t_2}, \tag{73}$$

where σ_{y2A} is the normal stress in the y_2-direction for $y_2 = -b/2$ and t_2 is the thickness of the beam component 2.

The first part of the expressions for stresses represent the stresses for the case of rigid cross-sections. The second part ($\Delta\sigma$ and ΔT) represent the additional stresses due to cross-section distortion, defined by the load q_{zcs} and internal forces Q_{zcs} and M_{ycs}, and the moment m_{ycs} (see Appendix A).

The same holds for the displacements given by (53) and (54), and (63) and (64), where the additional displacements due to distortion are defined by the displacements w_{1-3} and β_{1-3}, and the moment M_{ycs} (see Appendix A).

3.6. Boundary conditions

The boundary conditions at the ends of a beam may be formulated as follows

$$w_1 = w_1^*, \quad w_3 = w_3^*, \quad \beta_1 = \beta_1^*, \quad \beta_3 = \beta_3^*. \tag{74}$$

Then, according to the first equation of (41),

$$w_{1-3} = w_1^* - w_3^*, \quad \beta_{1-3} = \beta_1^* - \beta_3^*, \tag{75}$$

and according to (56)

$$w = \frac{1}{2}(w_1^* + w_3^*) + \frac{I'_{yc} - I'_{ys}}{I_y}(w_1^* - w_3^*), \quad \beta = \frac{1}{2}(\beta_1^* + \beta_3^*) + \frac{I'_{yc} - I'_{ys}}{I_y}(\beta_1^* - \beta_3^*), \tag{76}$$

and according to (27)

$$\begin{aligned} u_2 &= -(\beta_1^* + \beta_3^*)\frac{h}{4} + \frac{b^2}{6EA_2k}[T_A - T_B - k(T_A + T_B)], \\ T_A - T_B - k(T_A + T_B) &= -(\beta_1^* - \beta_3^*)\frac{GA_2h}{b^2}. \end{aligned} \tag{77}$$

Thus, the boundary conditions are defined by the components w_1^*, w_3^* and β_1^*, β_3^*.

In terms of forces, the boundary conditions may be formulated as follows

$$Q_{z1} = Q_{z1}^*, \quad Q_{z3} = Q_{z3}^*, \quad M_{y1} = M_{y1}^*, \quad M_{y3} = M_{y3}^*. \tag{78}$$

Then, according to (59) and (46)

$$Q_{zcs} = \frac{Q_{z1}^* I_{ys} - Q_{z3}^* I_{yc}}{I_{yc} + I_{ys}}, \quad M_{ycs} = \left(\frac{M_{y1}^*}{I_{y1}} - \frac{M_{y3}^*}{I_{y3}}\right)I_{ycs}, \tag{79}$$

and from (58) and (48),

$$Q_z = 2(Q_{z1}^* + Q_{z3}^*), \quad M_y = 2\left(\frac{I'_{yc}}{I_{y1}} M_{y1}^* + \frac{I'_{ys}}{I_{y3}} M_{y3}^*\right). \tag{80}$$

Here the boundary conditions are defined by the components Q_{z1}^*, Q_{z3}^* and M_{y1}^*, M_{y3}^*.

The "simply supported" ends may be defined by

$$w_{t1} = w_1 = w_{t3} = w_3 = 0, \quad M_{y1} = M_{y3} = 0. \tag{81}$$

Then, according to the first equation of (75) and the second equation of (79)

$$w_{1-3} = 0, \quad M_{ycs} = 0, \tag{82}$$

and according to the first equation of (76) and the second equation of (80)

$$w_t = w = 0, \quad M_y = 0. \tag{83}$$

Referring to (65), (67), (69) and (4), it follows

$$\sigma_{x1} = \sigma_{x3} = \sigma_{x2} = \sigma_x = 0. \tag{84}$$

For the "built-in end", it may be written

$$w_1 = w_{t1} = w_3 = w_{t3} = 0, \quad \beta_1 = \beta_3 = 0, \tag{85}$$

for the left end (A);

$$w_{t1} = w_1 + \frac{M_{y1}^{(B)} - M_{y1}^{(A)}}{GA_{s1}} = 0, \quad w_{t3} = w_3 + \frac{M_{y3}^{(B)} - M_{y3}^{(A)}}{GA_{s3}} = 0, \quad \beta_1 = \beta_3 = 0, \tag{86}$$

for the right end (B).

Then, according to (75),

$$w_{1-3} = 0, \quad \beta_{1-3} = 0, \tag{87}$$

and according to (76),

$$w = 0, \quad \beta = 0, \tag{88}$$

for the left end (A);

$$w_t = w + \frac{M_y^{(B)} - M_y^{(A)}}{GA_s} = 0. \tag{89}$$

for the right end (B). The "free end" may be defined as

$$Q_{z1} = 0, \quad Q_{z3} = 0, \quad M_{y1} = 0, \quad M_{y3} = 0. \tag{90}$$

Then, according to (79),

$$Q_{zcs} = 0, \quad M_{ycs} = 0, \tag{91}$$

and according to (80),

$$Q_z = 0, \quad M_y = 0. \tag{92}$$

Referring to (65), (67), (69) and (4), it follows

$$\sigma_{x1} = \sigma_{x3} = \sigma_{x2} = \sigma_x = 0, \quad T_{xs1} = T_{xs3} = T_{xz2} = T_{xs} = 0. \tag{93}$$

4. Illustrative examples

In order to illustrate the application of the expressions for the stresses and displacements, a thin-walled beam with the double symmetric three-cell closed cross-section with the constant thickness is considered (see Appendix A).

4.1. Geometrical properties and the material

$$h = 2b, \quad A = 16A_2, \quad I_y = \frac{32}{3}A_2b^2, \quad S^*_{yA} = \frac{1}{6}A_2b, \quad S^*_{yB} = \frac{5}{6}A_2b, \quad k = -\frac{2}{3},$$

$$I_{y1} = \frac{8}{3}A_2b^2, \quad I_{y3} = \frac{2}{3}A_2b^2, \quad I'_{yc} \approx 3.282A_2b^2, \quad I'_{ys} \approx 2.052A_2b^2,$$

$$I_{yc} = 3A_2b^2, \quad I_{ys} = \frac{7}{3}A_2b^2, \quad I_{ycs} \approx 0.923A_2b^2,$$

$$k_\beta = 2GA_2, \quad \frac{E}{G} = 2.6, \quad v = 0.456\frac{l}{b},$$

$$\frac{I_y}{I_{ycs}} \approx 11.557, \quad \frac{2I'_{yc}}{I_{ycs}} \approx 7.112, \quad \frac{2I'_{ys}}{I_{ycs}} \approx 4.446, \quad \frac{I'_{yc} + I'_{ys}}{I_{ycs}} \approx 5.779,$$

$$\frac{I'_{yc} - I'_{ys}}{I_{ycs}} \approx 1.333, \quad \frac{2I'_{yc}}{I_y} \approx 0.615, \quad \frac{2I'_{ys}}{I_y} \approx 0.385.$$

Here l is the length of the beam. The shear areas are calculated as pure geometrical properties, for Poisson‘s ratio equal to zero (Bhat and de Oliveira, 1985; Cowper, 1966)

$$A_s = AK, \quad K = 0.424, \quad A_s = AK = 6.784A_2.$$

4.2. Simply supported beam under uniformly distributed load

The internal forces at $x = 0$ (Example 1, Appendix A)

$$Q_{zcs} = \frac{q_{zcs}l}{2} = \frac{q_zl}{2} \times \frac{q_{zcs}}{q_z} = Q_z\frac{q_{zcs}}{q_z},$$

$$Q_{zcs} - m_{ycs} = \frac{q_{zcs}l}{2}\chi_3 = Q_z\frac{q_{zcs}}{q_z}\chi_3,$$

$$m_{ycs} = Q_z\frac{q_{zcs}}{q_z}(1 - \chi_3).$$

The internal forces and displacements at $x = l/2$ (Example 1, Appendix A):

$$M_{ycs} = \frac{q_{zcs}l^2}{8}\chi_0 = \frac{q_zl^2}{8} \times \frac{q_{zcs}}{q_z}\chi_0 = M_y\frac{q_{zcs}}{q_z}\chi_0,$$

$$w_{1-3} = \frac{5q_{zcs}l^4}{384EI_{ycs}}\varphi_0 = 11.557\frac{5q_zl^4}{384EI_y} \times \frac{q_{zcs}}{q_z}\varphi_0 = 11.557w\frac{q_{zcs}}{q_z}\varphi_0 = 11.557w_t\frac{q_{zcs}}{q_z} \times \frac{\varphi_0}{\eta_0},$$

$$w_t = \eta_0 w, \quad \eta_0 = 1 + \frac{48EI_y}{5GA_sl^2} = 1 + 39.245\left(\frac{b}{l}\right)^2.$$

Additional stresses due to distortion at $x = 0$:

$$\Delta T_{xs1} = 4.446\frac{Q_{zcs} - m_{ycs}}{I_y}S^*_{y1} - 1.333\frac{Q_{zcs}}{I_y}S^*_{yA}\Big|_A^{A'} + \frac{m_{ycs}}{2b}\Big|_A^{A'}$$

$$= 4.446\frac{Q_z}{I_y}S^*_{y1}\frac{q_{zcs}}{q_z}\chi_3(v) - 1.333\frac{Q_z}{I_y}S^*_{yA}\Big|_A^{A'}\frac{q_{zcs}}{q_z} + 5.333\frac{Q_z}{I_y}A_2 b\frac{q_{zcs}}{q_z}(1-\chi_3)\Big|_A^{A'},$$

$$\Delta T_{xs3} = -7.112\frac{Q_{zcs} - m_{ycs}}{I_y}S^*_{y3} - 1.333\frac{Q_{zcs}}{I_y}S^*_{yB} - \frac{m_{ycs}}{2b}$$

$$= -7.112\frac{Q_z}{I_y}S^*_{y3}\frac{q_{zcs}}{q_z}\chi_3 - 1.333\frac{Q_z}{I_y}S^*_{yB}\frac{q_{zcs}}{q_z} - 5.333\frac{Q_z}{I_y}A_2 b\frac{q_{zcs}}{q_z}(1-\chi_3),$$

$$\Delta T_{xz2} = 0.667\frac{Q_{zcs}}{I_y}(S^*_{yA} - S^*_{yB}) - 1.333\frac{Q_{zcs}}{I_y}(S^*_{yA} + S^*_{yB})\frac{z_2}{b} - \frac{m_{ycs}}{2b}$$

$$= 0.667\frac{Q_z}{I_y}(S^*_{yA} - S^*_{yB})\frac{q_{zcs}}{q_z} - 1.333\frac{Q_z}{I_y}(S^*_{yA} + S^*_{yB})\frac{z_2}{b}\times\frac{q_{zcs}}{q_z} - 5.333\frac{Q_z}{I_y}A_2 b\frac{q_{zcs}}{q_z}(1-\chi_3).$$

Additional stresses and displacements due to distortion at $x = l/2$:

$$\Delta\sigma_{x1} = 4.446\frac{M_{ycs}}{I_y}z_1 = 4.446\frac{M_y}{I_y}z_1\frac{q_{zcs}}{q_z}\chi_0,$$

$$\Delta\sigma_{x3} = -7.112\frac{M_{ycs}}{I_y}z_3 = -7.112\frac{M_y}{I_y}z_3\frac{q_{zcs}}{q_z}\chi_0,$$

$$\Delta\sigma_{x2} = 1.333\frac{M_{ycs}}{I_y}b - 2.223\frac{M_{ycs}}{I_y}b + 5.779\frac{M_{ycs}}{I_y}2z_2 + +2.223\frac{M_{ycs}}{I_y}b\left[\frac{3}{2}\left(\frac{2z_2}{b}\right)^2 - \frac{1}{2}\right]$$

$$= 1.333\frac{M_y}{I_y}b\frac{q_{zcs}}{q_z}\chi_0 - 2.223\frac{M_y}{I_y}b\frac{q_{zcs}}{q_z}\chi_0 + 5.779\frac{M_y}{I_y}2z_2\frac{q_{zcs}}{q_z}\chi_0 + 2.223\frac{M_y}{I_y}b\frac{q_{zcs}}{q_z}\left[\frac{3}{2}\left(\frac{2z_2}{b}\right) - \frac{1}{2}\right]\chi_0,$$

$$\Delta w_1 = 0.385 w_{1-3} = 4.446 w\frac{q_{zcs}}{q_z}\varphi_0 = 4.446 w_t\frac{q_{zcs}}{q_z}\times\frac{\varphi_0}{\eta_0},$$

$$\Delta w_3 = -0.615 w_{1-3} = -7.112 w\frac{q_{zcs}}{q_z}\varphi_0 = -7.112 w_t\frac{q_{zcs}}{q_z}\times\frac{\varphi_0}{\eta_0},$$

$$\Delta w_{s1} = 4.446\frac{M_{ycs} - M^{(A)}_{ycs}}{GA_s} = 4.446 w_s\frac{q_{zcs}}{q_z}\chi_0 = 4.446 w_t\frac{q_{zcs}}{q_z}\times\frac{\chi_0}{\eta'_0},$$

$$\Delta w_{s3} = -7.112\frac{M_{ycs} - M^{(A)}_{ycs}}{GA_s} = -7.112 w_s\frac{q_{zcs}}{q_z}\chi_0 = -7.112 w_t\frac{q_{zcs}}{q_z}\times\frac{\chi_0}{\eta_0},$$

$$w_t = \frac{w_s}{\eta'_0},\quad \eta'_0 = \frac{\frac{48EI_y}{5GA_s l^2}}{1 + \frac{48EI_y}{5GA_s l^2}} = \frac{39.24\left(\frac{b}{l}\right)^2}{1 + 39.24\left(\frac{b}{l}\right)^2}.$$

4.3. Built-in beam under uniformly distributed load

Components of the internal forces at $x = 0$ (Example 2, Appendix A):

$$M_{y\text{cs}} = -\frac{q_{z\text{cs}} l^2}{12}\chi_2 = -\frac{q_z l^2}{12} \times \frac{q_{z\text{cs}}}{q_z}\chi_2 = M_y \frac{q_{z\text{cs}}}{q_z}\chi_2,$$

$$Q_{z\text{cs}} = \frac{q_{z\text{cs}} l}{2} = \frac{q_z l}{2} \times \frac{q_{z\text{cs}}}{q_z} = Q_z \frac{q_{z\text{cs}}}{q_z}.$$

Components of the internal forces and displacements at $x = l/2$ (Example 2, Appendix A):

$$M_{y\text{cs}} = \frac{q_{z\text{cs}} l^2}{24}\chi_1 = \frac{q_z l^2}{24} \times \frac{q_{z\text{cs}}}{q_z}\chi_1 = M_y \frac{q_{z\text{cs}}}{q_z}\chi_1,$$

$$w_{1-3} = \frac{q_{z\text{cs}} l^4}{384 E I_{y\text{cs}}}\varphi_1 = 11.557 \frac{q_z l^4}{384 E I_y} \times \frac{q_{z\text{cs}}}{q_z}\varphi_1 = 11.557 w \frac{q_{z\text{cs}}}{q_z}\varphi_1$$

$$= 11.557 w_{\text{t}} \frac{q_{z\text{cs}}}{q_z} \times \frac{\varphi_1}{\eta_1}, \quad w_{\text{t}} = \eta_1 w, \quad \eta_1 = 1 + \frac{48 E I_y}{G A_{\text{s}} l^2} = 1 + 196.2\left(\frac{b}{l}\right)^2.$$

Additional stresses due to distortion at $x = 0$:

$$\Delta\sigma_{x1} = 4.446 \frac{M_{y\text{cs}}}{I_y} z_1 = 4.446 \frac{M_y}{I_y} z_1 \frac{q_{z\text{cs}}}{q_z}\chi_2,$$

$$\Delta T_{xs1} = 4.446 \frac{Q_{z\text{cs}}}{I_y} S^*_{y1} - 1.333 \frac{Q_{z\text{cs}}}{I_y} S^*_{yA}\Big|_A^{A'} = 4.446 \frac{Q_z}{I_y} S^*_{y1} \frac{q_{z\text{cs}}}{q_z} - 1.333 \frac{Q_z}{I_y} S^*_{yA}\Big|_A^{A'} \frac{q_{z\text{cs}}}{q_z},$$

$$\Delta\sigma_{x3} = -7.112 \frac{M_{y\text{cs}}}{I_y} z_3 = -7.112 \frac{M_y}{I_y} z_3 \frac{q_{z\text{cs}}}{q_z}\chi_2,$$

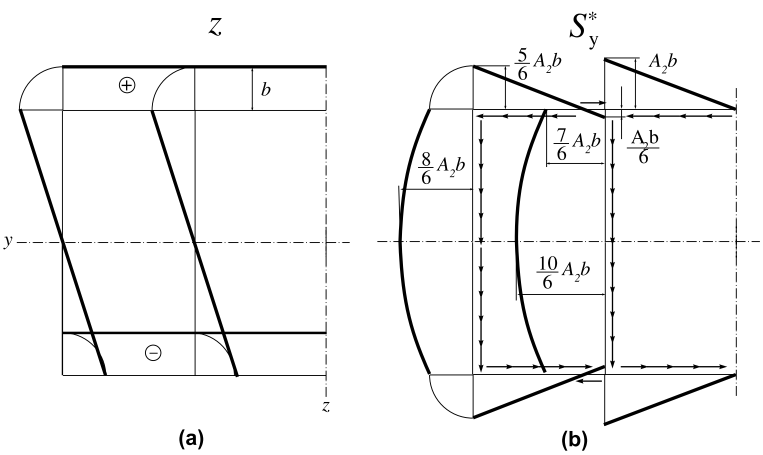

Fig. 4. Geometrical properties of the rigid cross-section: (a) z-coordinate; (b) S_y^*-coordinate (statical moment of the cut-off portion of cross-section area).

$$\Delta T_{xs3} = -7.112\frac{Q_{zcs}}{I_y}S^*_{y3} - 1.333\frac{Q_{zcs}}{I_y}S^*_{yB} = -7.112\frac{Q_z}{I_y}S^*_{y3}\frac{q_{zcs}}{q_z} - 1.333\frac{Q_z}{I_y}S^*_{yB}\frac{q_{zcs}}{q_z},$$

$$\begin{aligned}\Delta\sigma_{x2} &= 1.333\frac{M_{ycs}}{I_y}b - 2.223\frac{M_{ycs}}{I_y}b + 5.779\frac{M_{ycs}}{I_y}2z_2 + 2.223\frac{M_{ycs}}{I_y}b\left[\frac{3}{2}\left(\frac{2z_2}{b}\right)^2 - \frac{1}{2}\right]\\ &= 1.333\frac{M_y}{I_y}b\frac{q_{zcs}}{q_z}\chi_2 - 2.223\frac{M_y}{I_y}b\frac{q_{zcs}}{q_z}\chi_2 + 5.779\frac{M_y}{I_y}2z_2\frac{q_{zcs}}{q_z}\chi_2 + 2.223\frac{M_y}{I_y}b\frac{q_{zcs}}{q_z}\left[\frac{3}{2}\left(\frac{2z_2}{b}\right) - \frac{1}{2}\right]\chi_2,\end{aligned}$$

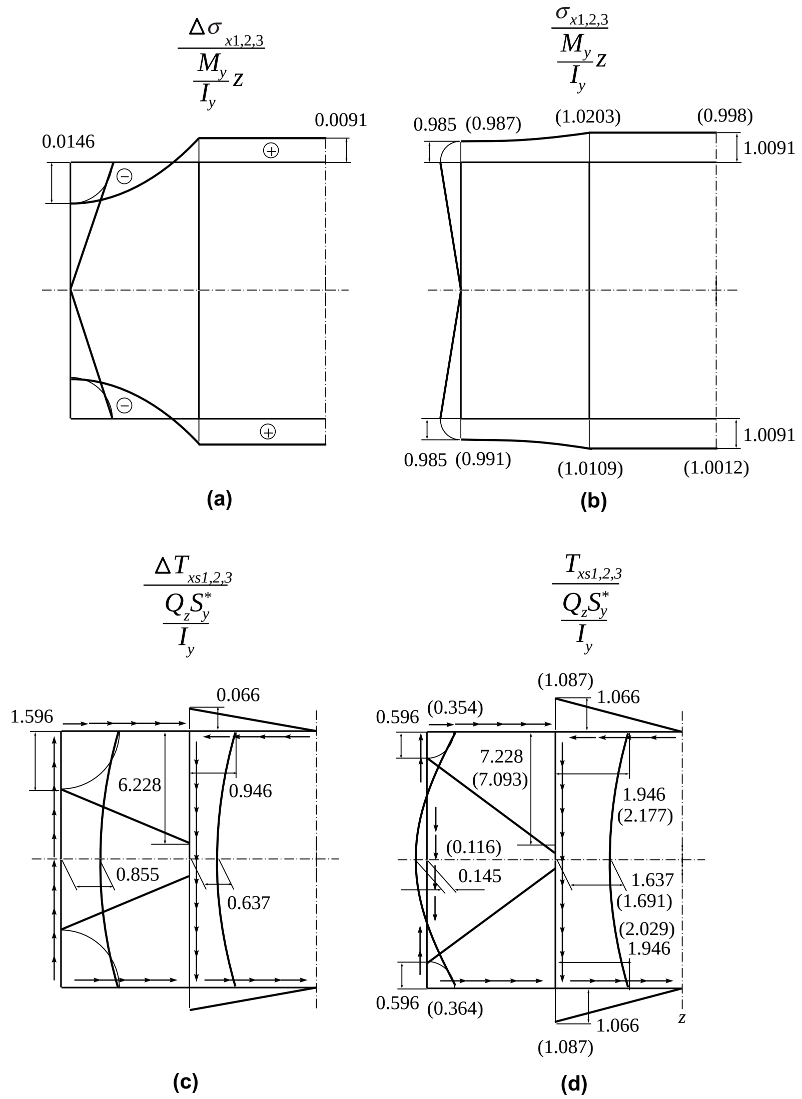

Fig. 5. Stresses for the simply supported beam: (a) additional normal stresses due to distortion at $x = l/2$; (b) normal stresses due to bending with distortion at $x = l/2$ (FEM in brackets); (c) additional shear stresses due to distortion at $x = 0$; (d) shear stresses due to bending with distortion at $x = 0$.

$$\Delta T_{xz2} = 1.333\frac{Q_{zcs}}{2I_y}(S^*_{yA} - S^*_{yB}) - 1.333\frac{Q_{zcs}}{I_y}(S^*_{yA} + S^*_{yB})\frac{z_2}{b}$$
$$= 0.667\frac{Q_z}{I_y}(S^*_{yA} - S^*_{yB})\frac{q_{zcs}}{q_z} - 1.333\frac{Q_z}{I_y}(S^*_{yA} + S^*_{yA})\frac{z_2}{b} \times \frac{q_{zcs}}{q_z}.$$

Additional stresses and displacements due to distortion at $x = l/2$:

$$\Delta\sigma_{x1} = 4.446\frac{M_{ycs}}{I_y}z_1 = 4.446\frac{M_y}{I_y}z_1\frac{q_{zcs}}{q_z}\chi_1,$$
$$\Delta\sigma_{x3} = -7.112\frac{M_{ycs}}{I_y}z_3 = -7.112\frac{M_y}{I_y}z_3\frac{q_{zcs}}{q_z}\chi_1,$$
$$\Delta w_1 = 0.385w_{1-3} = 4.446w\frac{q_{zcs}}{q_z}\varphi_1 = 4.446w_t\frac{q_{zcs}}{q_z} \times \frac{\varphi_1}{\eta_1},$$
$$\Delta w_3 = -0.615w_{1-3} = -7.112w\frac{q_{zcs}}{q_z}\varphi_1 = -7.112w_t\frac{q_{zcs}}{q_z} \times \frac{\varphi_1}{\eta_1},$$
$$\Delta w_{s1} = 4.446\frac{M_{ycs} - M^{(A)}_{ycs}}{GA_s} = 4.446w_s\frac{q_{zcs}}{q_z}(\chi_1 + \chi_2) = 4.446w_t\frac{q_{zcs}}{q_z}(\chi_1 + \chi_2)\eta'_1,$$
$$\Delta w_{s3} = -7.112\frac{M_{ycs} - M^{(A)}_{ycs}}{GA_s} = -7.112w_s\frac{q_{zcs}}{q_z}(\chi_1 + \chi_2) = -7.112w_t\frac{q_{zcs}}{q_z}(\chi_1 + \chi_2)\eta'_1,$$
$$w_t = \frac{w_s}{\eta'_1}, \quad \eta'_1 = \frac{\frac{48EI_y}{GA_s l^2}}{1 + \frac{48EI_y}{GA_s l^2}} = \frac{196.2\left(\frac{b}{l}\right)^2}{1 + 196.2\left(\frac{b}{l}\right)^2}.$$

The additional stresses and displacements due to distortion are calculated for $l/b = 32$, i.e. $v = 14.59$ ($\varphi_0 = 0.0112$, $\varphi_1 = 0.0564$, $\chi_0 = 0.0094$, $\chi_1 = 0.0282$, $\chi_2 = 0.1915$, $\chi_3 = 0.0685$, $\psi_0 = 0.0131$); $\eta_0 = 1.0383$, $\eta'_0 = 0.0369$, $\eta_1 = 1.192$, $\eta'_1 = 0.161$.

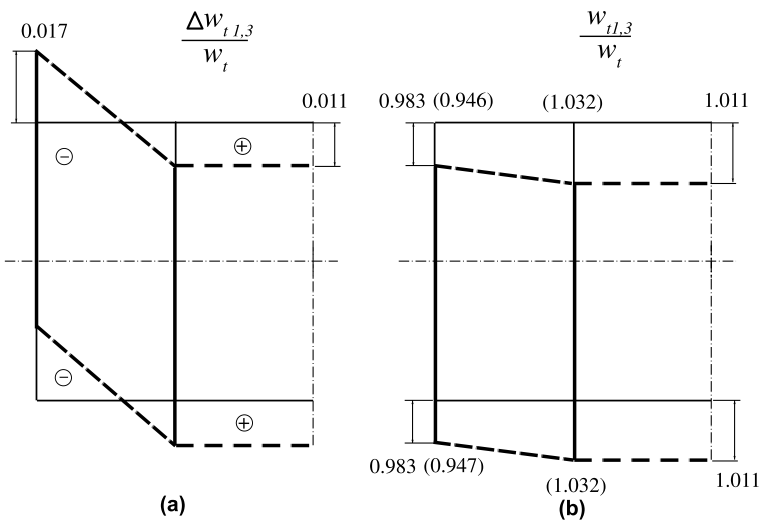

Fig. 6. Displacements for the simply supported beam at $x = l/2$: (a) additional displacements due to distortion; (b) displacements due to bending with distortion (FEM in brackets).

The stresses and displacements are calculated for the beam subjected to uniformly distributed load $q_{z1} = q_z/2$, $q_{z3} = 0$ ($q_{zcs}/q_z = 7/32$), both for simply supported and built-in beam ends, for $x = l/2$ and $x = 0$. The results are given in respect to the stresses and displacements for rigid cross-sections (Figs. 4–8).

In order to check accuracy of the obtained results, the normal stresses and displacements at $x = l/2$ for the simply supported beam are also calculated by the finite element method; the shear stresses are calculated for $x = 0$. The 3D membrane model of one-quarter of the beam is investigated. A high mesh density with 3200 quadrilateral elements and 12880 nodes is used. The displacements of the nodes are restricted at $x = 0$

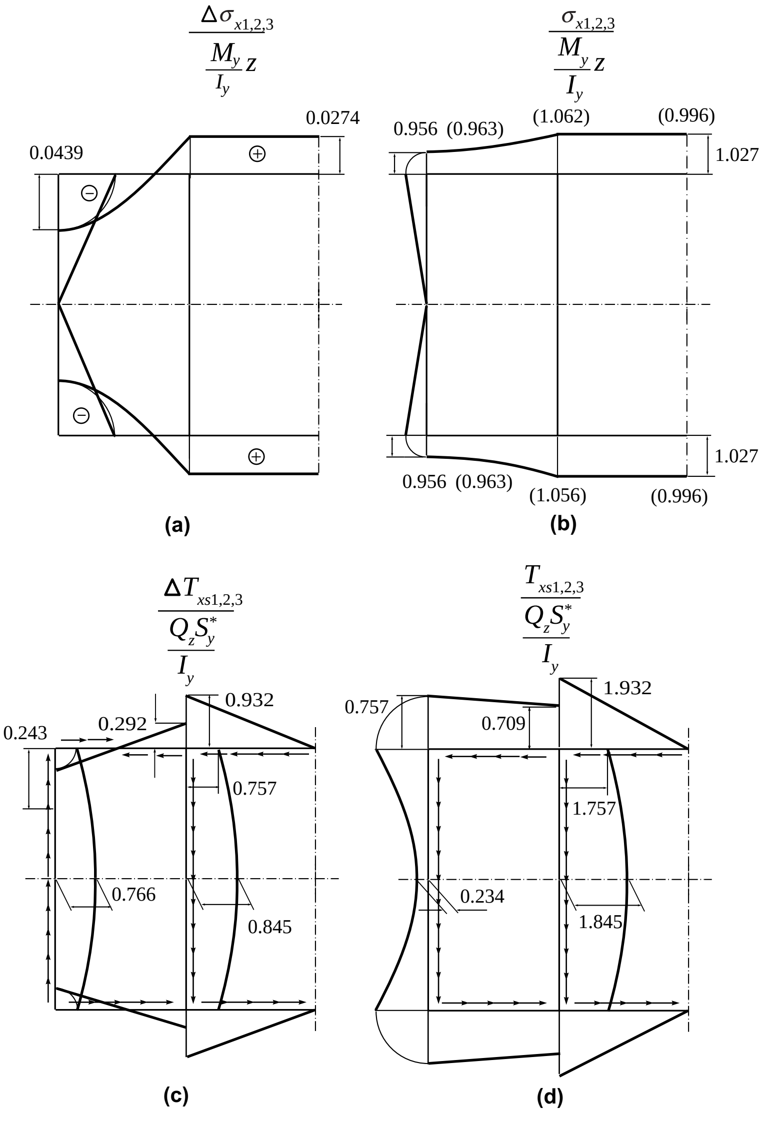

Fig. 7. Stresses for the built-in beam: (a) additional normal stresses due to distortion at $x = l/2$; (b) normal stresses due to bending with distortion at $x = l/2$; (c) additional shear stresses due to distortion at $x = 0$; (d) shear stresses due to bending with distortion at $x = 0$ (FEM in brackets).

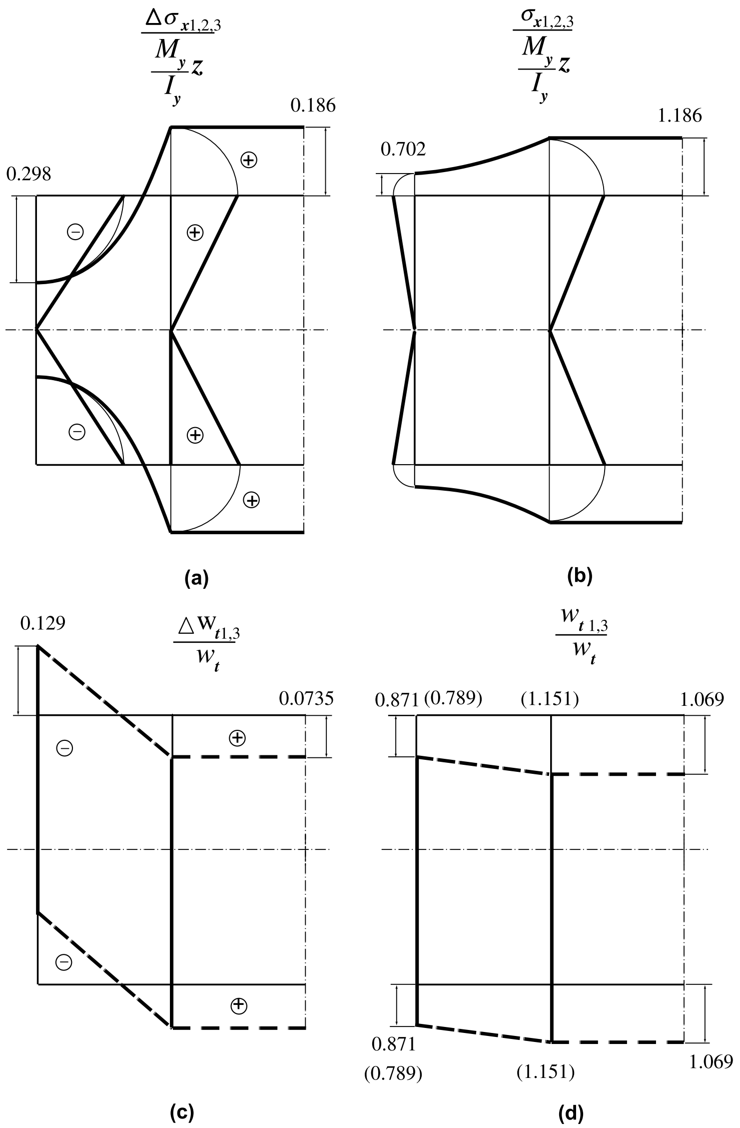

Fig. 8. Stresses and displacements for the built-in beam: (a) additional normal stresses at $x = 0$; (b) stresses due to bending with distortion at $x = 0$; (c) additional displacements due to distortion at $x = l/2$; (d) displacements due to bending with distortion at $x = l/2$ (FEM in brackets).

in the cross-section contour direction and at $x = l/2$ in the beam longitudinal direction; at $y = 0$ (in the plane of symmetry) the displacements are restricted in the beam transverse direction.

5. Conclusion

An analytical method has been applied to estimate the additional stresses and displacements due to distortion of the cross-sections of thin-walled beams subjected to bending. Simple rectangular cross-sections with three and two closed cells with double axes of symmetry are considered. It is assumed that beams are long enough that warping due to shearing may be ignored in the stress calculation (in the case of rigid cross-sections, i.e. when the cross-sections maintain their shape). The cross-section distortion is considered in the limit, by assuming that beam walls are hinged along their longitudinal edges.

The additional stresses and displacements due to the cross-section distortion are given in the analytical closed form and compared to the stresses and displacements of the ordinary bending theory (where the cross-sections maintain their shape).

It is shown that the additional stresses and displacements due to the cross-section distortion can be significant (compared to the stresses and displacements of the ordinary bending theory), particularly, the additional shear stresses.

A typical cross-section with three cells is analysed, where the ratio of the beam length and the cross-section breadth was equal 8 (the beam length to the cross-section height ratio equal 16) is analysed. The loads were distributed along the inner vertical walls only. The ends of the beam were simply supported and built-in, respectively.

The comparison for the simply supported beam under uniformly distributed load along the inner vertical walls to the finite element solution of the problem has shown acceptable agreements of obtained results.

Although the hinges between plates do not occur in actuality, it is important to analyse such conditions, together with the ordinary beam theory (where the cross-section is assumed rigid), as limits of the actual behaviour of the structure. In fact, there are no enough stiff cross-sections in actual thin-walled beam structures to guarantee the cross-section shape, especially under nonuniform load distributions in the transverse direction.

The real stiffness of the cross-section structure can be easily included in the consideration. The torsional stiffness of the plates together with bending stiffness of transverse framing (if the structure is framed) and the shear stiffness of transverse bulkheads (if there are any) should be taken into account.

The same approach may be used in the case of thin-walled curved beams. The cross-sections of the components beams may be approximate by rectangular cross-sections; or be considered as curvilinear cross-sections. In the compatibility conditions and equilibrium equations, the cross-section properties of the beam components will be changed only. It should be noted that relative vertical displacements are small, in comparison with vertical displacement of the beams. In that case, in the finite element analysis the shell elements must be used.

Acknowledgments

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Appendix A

The solution of Eq. (41) may be presented, by the method of initial parameters, as follows (Pavazza, 1991)

$$\mathbf{v} = \mathbf{K}\mathbf{v}_0 + \mathbf{I}, \tag{A.1}$$

where

$$\mathbf{v} = [\,Q_{zcs} \quad M_{ycs} \quad \beta_{1-3} \quad w_{1-3}\,]^{\mathrm{T}}, \quad \mathbf{v}_0 = [\,Q_{zcs}(0) \quad M_{ycs}(0) \quad \beta_{1-3}(0) \quad w_{1-3}(0)\,]^{\mathrm{T}},$$

$$\mathbf{K} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{l}{g}\,\mathrm{sh}\,\frac{g}{l}x & \mathrm{ch}\,\frac{g}{l}x & EI_{ycs}\frac{g}{l}\,\mathrm{sh}\,\frac{g}{l}x & 0 \\ \frac{l^2}{EI_{ycs}}g^2\left(\mathrm{ch}\,\frac{g}{h}x-1\right) & \frac{l}{EI_{ycs}g}\,\mathrm{sh}\,\frac{g}{l}x & \mathrm{ch}\,\frac{g}{l}x & 0 \\ -\frac{l^2}{EI_{ycs}g^2}\left(\frac{l}{g}\,\mathrm{sh}\,\frac{g}{l}x-x\right) & -\frac{l^2}{EI_{ycs}g^2}\left(\mathrm{ch}\,\frac{g}{l}x-1\right) & -\frac{l}{g}\,\mathrm{sh}\,\frac{g}{l}x & 1 \end{bmatrix},$$

$$I = \left[-EI_{ycs}\frac{\mathrm{d}^3 w_{(1-3)p}}{\mathrm{d}x^3} + EI_{ycs}\frac{g^2}{l^2}\times\frac{\mathrm{d}w_{(1-3)p}}{\mathrm{d}x} \quad -EI_{ycs}\frac{\mathrm{d}^2 w_{(1-3)p}}{\mathrm{d}x^2} - \frac{\mathrm{d}w_{(1-3)p}}{\mathrm{d}x} \quad w_{(1-3)p}\right]^{\mathrm{T}}, \tag{A.2}$$

where

$$w_{(1-3)p} = \frac{1}{EI_{ycs}}\times\frac{l^2}{g^2}\int_0^x\left[\frac{l}{g}\,\mathrm{sh}\,\frac{g}{l}(x-\xi)-(x-\xi)\right]q_{zcs}\mathrm{d}\xi, \quad g = l\sqrt{\frac{k_\beta}{EI_{ycs}}}. \tag{A.3}$$

For the uniformly distributed load, $q_{zcs} = q_{cs}$:

$$\begin{aligned} w_{(1-3)p} &= \frac{q_{cs}l^2}{EI_{ycs}g^2}\left[\frac{l^2}{g^2}\left(\mathrm{ch}\,\frac{g}{l}x-1\right)-\frac{x^2}{2}\right], \\ \mathbf{I} &= \left[-q_{cs}x \quad -q_{cs}\frac{l^2}{g^2}\left(\mathrm{ch}\,\frac{g}{l}x-1\right) \quad -\frac{q_{cs}l^2}{EI_{ycs}g^2}\left(\frac{l}{g}\,\mathrm{sh}\,\frac{g}{l}x-x\right)\frac{q_{cs}l^2}{EI_{ycs}g^2}\left[\frac{l^2}{g^2}\left(\mathrm{ch}\,\frac{g}{l}x-1\right)-\frac{x^2}{2}\right]\right]^{\mathrm{T}}. \end{aligned} \tag{A.4}$$

In the case of many load distribution fields, it may be written

$$\mathbf{v} = \mathbf{K}\mathbf{v}_0 + \mathbf{I} + \sum_{r=2}^{i}(\mathbf{K}_r\mathbf{v}_{a_{r-1},r} + \mathbf{I}_r), \quad (i = 2, 3, \ldots, m), \tag{A.5}$$

where

$$\mathbf{v}_{a_{r-1},r} = [\,Q_{zcs,a_{r-1},r} \quad M_{ycs,a_{r-1},r} \quad \beta_{(1-3)a_{r-1},r} \quad w_{(1-3)a_{r-1},r}\,]^{\mathrm{T}},$$

$$\mathbf{K}_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{l}{g}\,\mathrm{sh}\,\frac{g}{l}\tilde{x} & \mathrm{ch}\,\frac{g}{l}\tilde{x} & EI_{ycs}\frac{g}{l}\,\mathrm{sh}\,\frac{g}{l}\tilde{x} & 0 \\ \frac{l^2}{EI_{ycs}g^2}\left(\mathrm{ch}\,\frac{g}{l}\tilde{x}-1\right) & \frac{l}{EI_{ycs}g}\,\mathrm{sh}\,\frac{g}{l}\tilde{x} & \mathrm{ch}\,\frac{g}{l}\tilde{x} & 0 \\ -\frac{l^2}{EI_{ycs}g^2}\left(\frac{l}{g}\,\mathrm{sh}\,\frac{g}{l}\tilde{x}-\tilde{x}\right) & -\frac{l^2}{EI_{ycs}g^2}\left(\mathrm{ch}\,\frac{g}{l}\tilde{x}-1\right) & -\frac{l}{g}\,\mathrm{sh}\,\frac{g}{l}\tilde{x} & 1 \end{bmatrix}, \tag{A.6}$$

$$\mathbf{I}_r = \left[-EI_{ycs}\frac{\mathrm{d}^3 w_{(1-3)p,r}}{\mathrm{d}x^3} + EI_{ycs}\frac{g^2}{l^2}\times\frac{\mathrm{d}w_{(1-3)p,r}}{\mathrm{d}x} \quad -EI_{ycs}\frac{\mathrm{d}^2 w_{(1-3)p,r}}{\mathrm{d}x^2} - \frac{\mathrm{d}w_{(1-3)p,r}}{\mathrm{d}x} \quad w_{(1-3)p,r}\right]^{\mathrm{T}}, \quad \tilde{x} = x - a_{r-1}.$$

For the partly distributed uniform load, along $0 \leqslant x \leqslant a_1$ ($r = 2$):

$$w_{(1-3)p,r,2} = -\frac{q_{cs}l^2}{EI_{ycs}g^2}\int_0^x\left[\frac{l}{g}\,\mathrm{sh}\,\frac{g}{l}(x-\xi)-(x-\xi)\right]d\xi = -\frac{q_{cs}l^2}{EI_{ycs}g^2}\left\{\frac{l^2}{g^2}\left[\mathrm{ch}\,\frac{g}{l}(x-a_1)-1\right]-\frac{(x-a_1)^2}{2}\right\},$$

$$\begin{aligned} \mathbf{I}_2 = \Bigg[& q_{cs}(x-a_1)q_{cs}\frac{l^2}{g^2}\left[\mathrm{ch}\,\frac{g}{l}(x-a_1)-1\right]\frac{q_{cs}l^2}{EI_yg^2}\left[\frac{l}{g}\,\mathrm{sh}\,\frac{g}{l}(x-a_1)-(x-a_1)\right] \\ & -\frac{q_{cs}l^2}{EI_{ycs}g^2}\left\{\frac{l^2}{g^2}\left[\mathrm{ch}\,\frac{g}{l}(x-a_1)-1\right]-\frac{(x-a_1)^2}{2}\right\}\Bigg]^{\mathrm{T}}. \end{aligned} \tag{A.7}$$

For the concentrated force, Q^*_{zcs}, at $x = a_{r-1}$:

$$\mathbf{v}_{a_{r-1,r}} = [-Q^*_{zcs} \quad 0 \quad 0 \quad 0]^{\mathrm{T}}. \tag{A.8}$$

Example 1. Simply supported beam under the uniformly distributed load; boundary conditions by (81):

$$
\begin{aligned}
&Q_{zcs} = \frac{q_{cs} l}{2}\left(1 - 2\frac{x}{l}\right),\\
&Q_{zcs} - m_{ycs} = \frac{q_{cs} l}{g}\psi \mathrm{sh} g\left(\frac{1}{2} - \frac{x}{l}\right),\\
&m_{ycs} = \frac{q_{cs} l}{g}\left[1 - 2\frac{x}{l} - \frac{2\psi}{g}\mathrm{sh} g\left(\frac{1}{2} - \frac{x}{l}\right)\right],\\
&M_{ycs} = \frac{q_{cs} l^2}{g^2}\left[1 - \psi \mathrm{ch} g\left(\frac{1}{2} - \frac{x}{l}\right)\right],\\
&\beta_{1-3} = -\frac{q_{cs} l^3}{2EI_{ycs} g^2}\left[1 - 2\frac{x}{l} - \frac{2\psi}{g}\mathrm{sh} g\left(\frac{1}{2} - \frac{x}{l}\right)\right],\\
&w_{1-3} = \frac{q_{cs} l^4}{2EI_{ycs} g^2}\left\{\frac{x}{l} - \left(\frac{x}{l}\right)^2 - \frac{2}{g^2}\left[1 - \psi \mathrm{ch} g\left(\frac{1}{2} - \frac{x}{l}\right)\right]\right\},
\end{aligned}
\tag{A.9}
$$

$$\psi = \frac{1}{\mathrm{ch}\frac{g}{2}},$$

$$
\begin{aligned}
&Q_{zcs}(0) = -Q_{zcs}(l) = \frac{ql}{2}, Q_{zcs}(0) - m_{ycs}(0) = -Q_{zcs}(l) + m_{ycs}(l) = \frac{q_{cs} l}{2}\chi_3(v),\\
&\beta_{1-3}(0) = -\beta_{1-3}(l) = -\frac{q_{cs} l^3}{24EI_{ycs}}\psi_0(v),\\
&M_{ycs}(l/2) = \frac{q_{cs} l^2}{8}\chi_0(v), \quad w_{1-3}(l/2) = \frac{5q_{cs} l^4}{384EI_{ycs}}\varphi_0(v),
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
&\chi_0(v) = \frac{2}{v^2}\left(1 - \frac{1}{\mathrm{ch}\, v}\right), \quad \varphi_0(v) = \frac{24}{5v^4}\left(\frac{v^2}{2} + \frac{1}{\mathrm{ch}\, v} - 1\right), \quad \psi_0(v) = \frac{3}{v^2}\left(1 - \frac{\mathrm{th}\, v}{v}\right),\\
&\chi_3(v) = \frac{\mathrm{th}\, v}{v}, \quad v = \frac{g}{2} = \frac{l}{2}\sqrt{\frac{k_\beta}{EI_{ycs}}}.
\end{aligned}
\tag{A.11}
$$

Example 2. Built-in beam under the uniformly distributed load; boundary conditions by (85):

$$
\begin{aligned}
&Q_{zcs} = \frac{q_{cs} l}{2}\left(1 - 2\frac{x}{l}\right),\\
&Q_{zcs} - m_{ycs} = \frac{q_{cs} l\psi}{g\chi}\mathrm{sh}\, g\left(\frac{1}{2} - \frac{x}{l}\right),\\
&m_{ycs} = \frac{q_{cs} l}{2}\left[1 - 2\frac{x}{l} - \frac{2\psi}{g\chi}\mathrm{sh}\, g\left(\frac{1}{2} - \frac{x}{l}\right)\right],\\
&M_{ycs} = \frac{q_{cs} l^2}{g^2}\left[1 - \frac{\psi}{\chi}\mathrm{ch}\, g\left(\frac{1}{2} - \frac{x}{l}\right)\right],\\
&\beta_{1-3} = -\frac{q_{cs} l^3}{2E_{ycs} g^2}\left[1 - 2\frac{x}{l} - \frac{2\psi}{g\chi}\mathrm{sh}\, g\left(\frac{1}{2} - \frac{x}{l}\right)\right],\\
&w_{1-3} = \frac{q_{cs} l^4}{2E_{ycs} g^2}\left\{\frac{x}{l} - \left(\frac{x}{l}\right)^2 - \frac{2}{g^2\chi}\left[1 - \psi \mathrm{ch}\, g\left(\frac{1}{2} - \frac{x}{l}\right)\right]\right\},
\end{aligned}
\tag{A.12}
$$

$$\chi = \frac{2}{g}\psi \operatorname{sh}\frac{g}{2},$$

$$Q_{zcs}(0) = -Q_{zcs}(l) = \frac{q_{cs}l}{2}, \quad M_{ycs}(0) = M_{ycs}(l) = -\frac{q_{cs}l^2}{12}\chi_2(v), \tag{A.13}$$

$$M_{ycs}(l/2) = \frac{q_{cs}l^2}{24}\chi_1(v), \quad w_{1-3}(l/2) = \frac{q_{cs}l^4}{384EI_{ycs}}\varphi_1(v),$$

$$\chi_1(v) = \frac{6}{v^2}\left(1 - \frac{v}{\operatorname{sh} v}\right), \quad \chi_2(v) = \frac{3}{v^2}\left(\frac{v}{\operatorname{th} v} - 1\right), \quad \varphi_1(v) = \frac{24}{v^3}\left(\frac{v}{2} - \operatorname{th}\frac{v}{2}\right). \tag{A.14}$$

Example 3. Simple supported beam under the concentrated force $Q^*_{zcs} = F$ at $x = l/2$; boundary conditions by (81):

$$Q_{zcs}(0) = -Q_{zcs}(l) = \frac{F}{2}, \quad Q_{zcs}(0) - m_{ycs}(0) = -Q_{zcs}(l) + m_{ycs}(l) = \frac{F}{2} \times \frac{1}{\operatorname{ch} v},$$

$$m_{ycs}(0) = -m_{ycs}(l) = \frac{F}{2}\left(1 - \frac{1}{\operatorname{ch} v}\right), \quad \beta_{1-3}(0) = -\beta_{1-3}(l) = -\frac{Fl^2}{16EI_{ycs}}\chi_0(v), \tag{A.15}$$

$$M_{ycs}(l/2) = \frac{Fl}{4}\chi_3(v), \quad w_{1-3}(l/2) = \frac{Fl^3}{48EI_{ycs}}\psi_0(v).$$

Example 4. Built-in beam under the concentrated force at $x = l/2$; boundary conditions given by (85):

$$Q_{zcs}(0) = -Q_{zcs}(l) = \frac{F}{2}, \quad M_{ycs}(0) = M_{ycs}(l) = -M_{ycs}(l/2) = -\frac{Fl}{8}\chi_4(v),$$

$$Q^l_{zcs}(l/2) = -Q^r_{zcs}(l/2) = \frac{F}{2}, \quad w_{1-3}(l/2) = \frac{Fl^3}{192EI_{ycs}}\varphi_1(v), \tag{A.16}$$

$$\chi_4(v) = \frac{2(\operatorname{ch} v - 1)}{v \operatorname{sh} v}. \tag{A.17}$$

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